Fourier’s Law and the Heat Equation

Chapter Two

Fourier's Law

- A rate equation that allows determination of the conduction heat flux from knowledge of the temperature distribution in a medium

- Its most general (vector) form for multidimensional conduction is:

$$\vec{q}'' = -k \nabla T$$

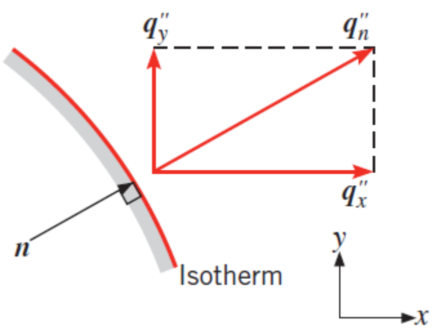
Implications:

– Heat transfer is in the direction of decreasing temperature (basis for minus sign).

– Fourier's law serves to define the thermal conductivity of the medium $\left(k \equiv -\vec{q}'' / \nabla T \right)$

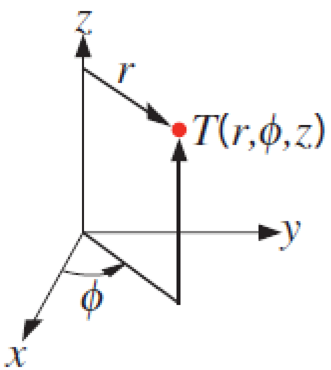
– Direction of heat transfer is perpendicular to lines of constant temperature (isotherms).

– Heat flux vector may be resolved into orthogonal components.



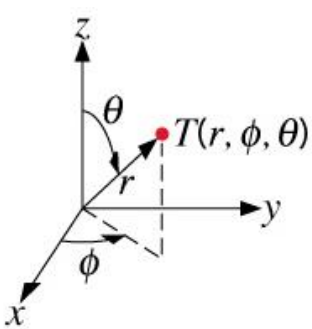
- Cartesian Coordinates: $T(x,y,z)$

$$\vec{q}'' = \underbrace{-k\frac{\partial T}{\partial x}}_{q''_x}\vec{i}\underbrace{-k\frac{\partial T}{\partial y}}_{q''_y}\vec{j}\underbrace{-k\frac{\partial T}{\partial z}}_{q''_z}\vec{k} \qquad (2.3)$$



- Cylindrical Coordinates: $T(r,\phi,z)$

$$\vec{q}'' = \underbrace{-k\frac{\partial T}{\partial r}}_{q''_r}\vec{i}\underbrace{-k\frac{\partial T}{r\partial \phi}}_{q''_\phi}\vec{j}\underbrace{-k\frac{\partial T}{\partial z}}_{q''_z}\vec{k} \qquad (2.24)$$



- Spherical Coordinates: $T(r,\phi,\theta)$

$$\vec{q}'' = \underbrace{-k\frac{\partial T}{\partial r}}_{q''_r}\vec{i}\underbrace{-k\frac{\partial T}{r\partial \theta}}_{q''_\theta}\vec{j}\underbrace{-k\frac{\partial T}{r\sin\theta\,\partial \phi}}_{q''_\phi}\vec{k} \qquad (2.27)$$

- In angular coordinates $(\phi \text{ or } \phi, \theta)$, the temperature gradient is still based on temperature change over a length scale and hence has units of °C/m and not °C/deg.

- Heat rate for one-dimensional, radial conduction in a cylinder or sphere:

  – Cylinder

  $$q_r = A_r q_r'' = 2\pi r L q_r'' \qquad \text{[W]}$$

  or,

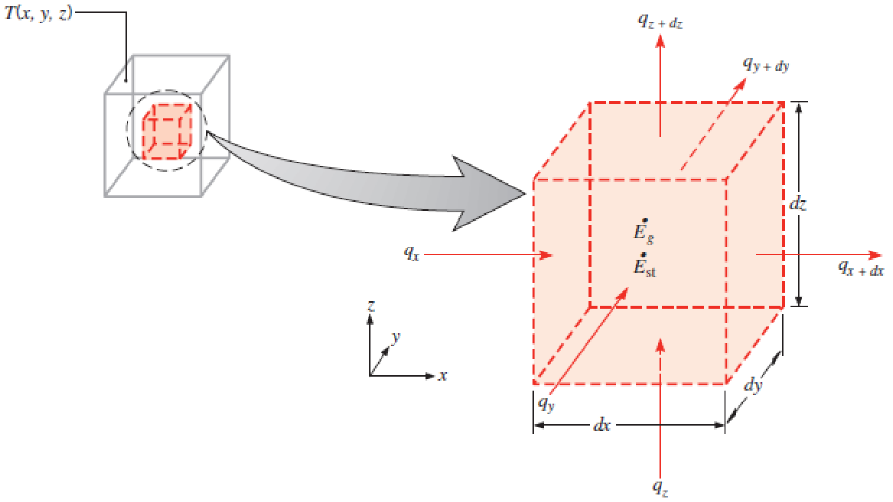
  $$q_r' = A_r' q_r'' = 2\pi r q_r'' \qquad \text{[W/m]}$$

  – Sphere

  $$q_r = A_r q_r'' = 4\pi r^2 q_r'' \qquad \text{[W]}$$

The Heat Equation

- A differential equation whose solution provides the temperature distribution in a stationary medium.
- Based on applying conservation of energy to a differential control volume through which energy transfer is exclusively by conduction.
- Cartesian Coordinates:



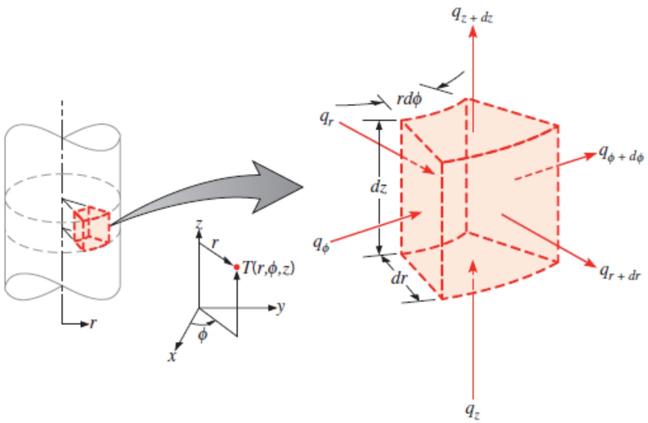
$$\underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)}+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \qquad (2.19)$$

Net transfer of thermal energy into the control volume (inflow-outflow)

Thermal energy generation

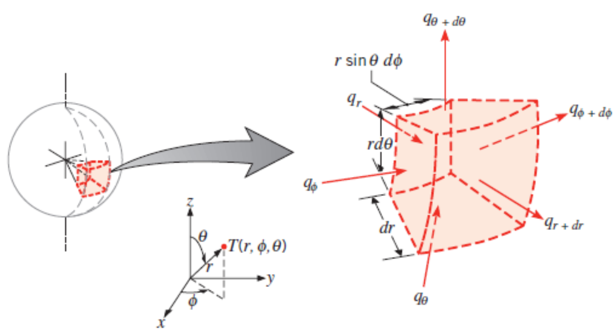
Change in thermal energy storage

- Cylindrical Coordinates:



$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \tag{2.26}$$

- Spherical Coordinates:



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \tag{2.29}$$

- One-Dimensional Conduction in a Planar Medium with Constant Properties and No Generation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)=\rho c_p\frac{\partial T}{\partial t}$$

becomes

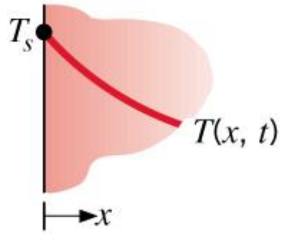
$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$\alpha \equiv \dfrac{k}{\rho c_p} \rightarrow$ thermal diffusivity of the medium $\left[\mathrm{m^2/s}\right]$

Boundary and Initial Conditions

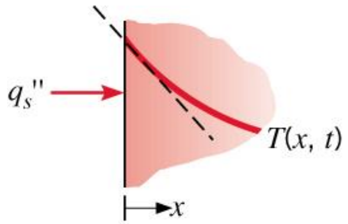
- For transient conduction, heat equation is first order in time, requiring specification of an initial temperature distribution: $T(x,t)_{t=0} = T(x,0)$
- Since heat equation is second order in space, two boundary conditions must be specified for each coordinate direction. Some common cases:

Constant Surface Temperature:



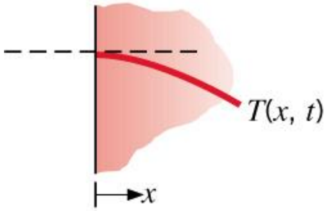
Constant Heat Flux:

Applied Flux



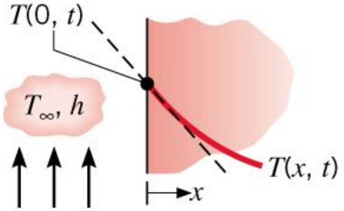
$$-k\frac{\partial T}{\partial x}\Big|_{x=0} = q_s''$$

Insulated Surface



$$\frac{\partial T}{\partial x}\Big|_{x=0} = 0$$

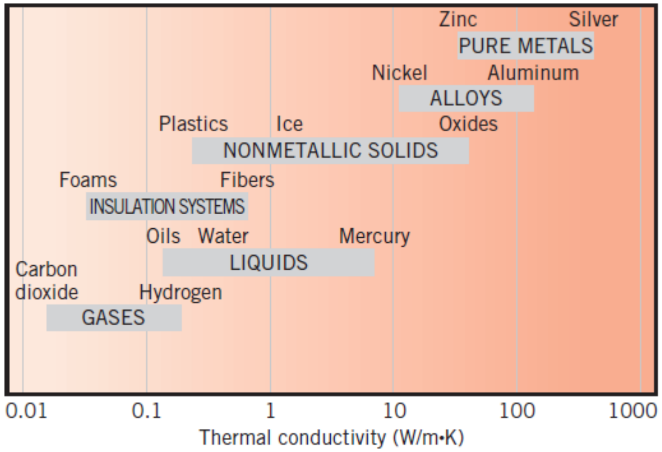
Convection:



$$-k\frac{\partial T}{\partial x}\Big|_{x=0} = h\left[T_\infty - T(0,t)\right]$$

Thermophysical Properties

Thermal Conductivity: A measure of a material's ability to transfer thermal energy by conduction.



Thermal Diffusivity: A measure of a material's ability to respond to changes in its thermal environment.

Property Tables:

- Solids: Tables A.1 – A.3
- Gases: Table A.4
- Liquids: Tables A.5 – A.7

Nanoscale Effects

- Conduction may be viewed as a consequence of energy carrier (electron or phonon) motion.

- For the solid state:

$$k = \frac{1}{3} C \bar{c} \lambda_{mfp} \tag{2.7}$$

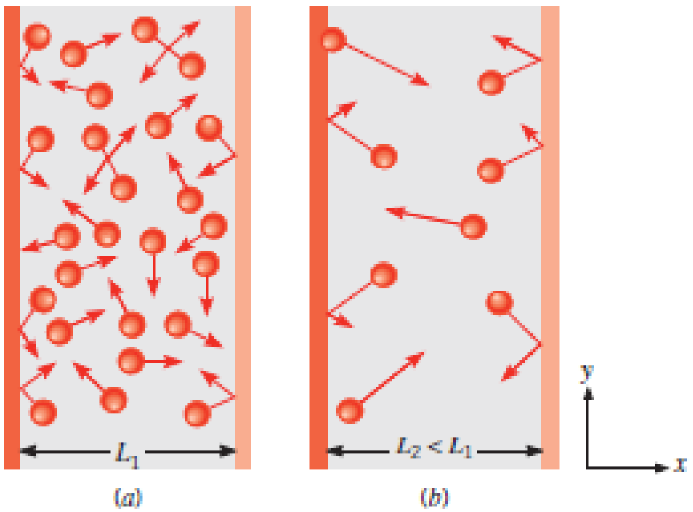
C: energy carrier specific heat per unit volume.

$\bar{c}$: average energy carrier velocity, $\bar{c} < \infty$.

λ_{mfp}: mean free path → average distance traveled by an energy carrier before a collision.

- Energy carriers also collide with physical boundaries, affecting their propagation.

  - External boundaries of a film of material. thick film (left) and thin film (right).

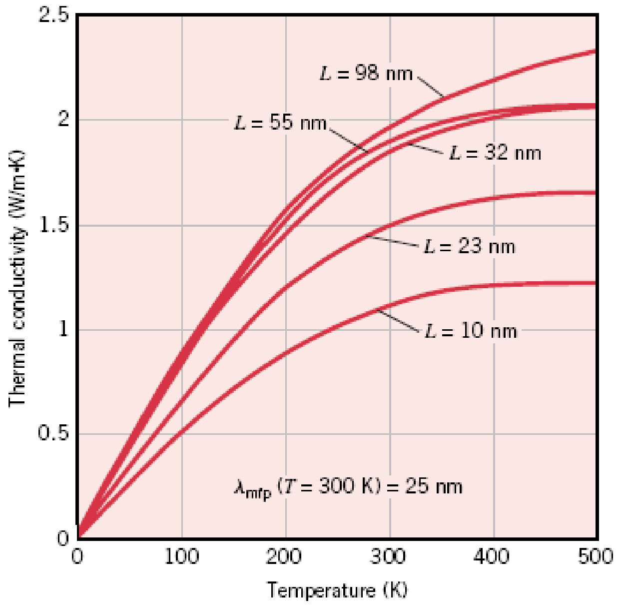


For $L/\lambda_{\text{mfp}} > 1$,

$$k_x / k = 1 - \lambda_{\text{mfp}} / (3L) \quad (2.9a)$$

$$k_y / k = 1 - 2\lambda_{\text{mfp}} / (3\pi L) \quad (2.9b)$$

where λ_{mfp} is the average distance traveled before experiencing a collision with another energy carrier or boundary (See Table 2.1 and Eq. 2.11).



➢ Grain boundaries within a solid

Measured thermal conductivity of a ceramic material vs. grain size, L. λ_{mfp} at $T \approx 300$ K $= 25$ nm.

- Fourier's law does not accurately describe the finite energy carrier propagation velocity. This limitation is not important except in problems involving extremely small time scales.

Typical Methodology of a Conduction Analysis

- Consider possible microscale or nanoscale effects in problems involving small physical dimensions or rapid changes in heat or cooling rates.

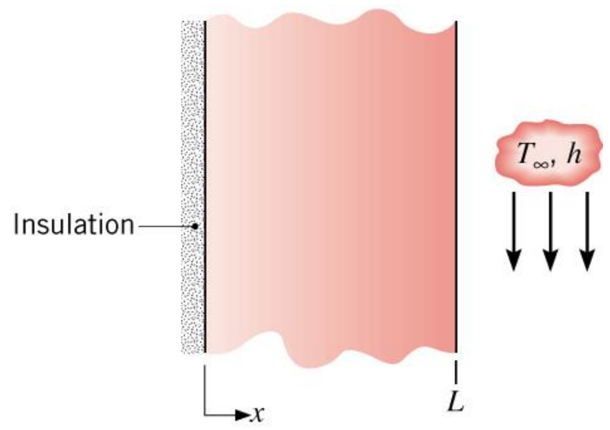
- Solve appropriate form of heat equation to obtain the temperature distribution.

- Knowing the temperature distribution, apply Fourier's law to obtain the heat flux at any time, location and direction of interest.

- Applications:

  Chapter 3: One-Dimensional, Steady-State Conduction
  Chapter 4: Two-Dimensional, Steady-State Conduction
  Chapter 5: Transient Conduction

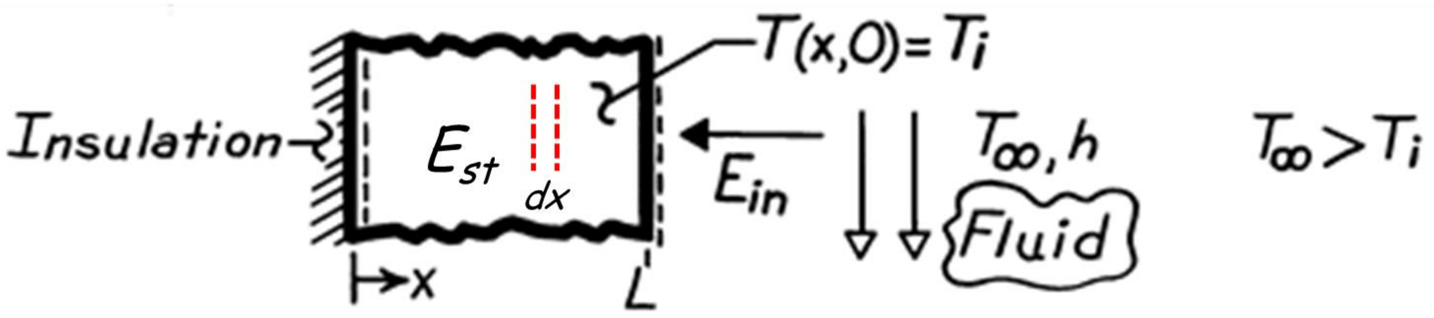
Problem 2.43 Thermal response of a plane wall to convection heat transfer.



KNOWN: Plane wall, initially at a uniform temperature, is suddenly exposed to convective heating.

FIND: (a) Differential equation and initial and boundary conditions which may be used to find the temperature distribution, $T(x,t)$; (b) Sketch $T(x,t)$ for the following conditions: initial ($t \leq 0$), steady-state ($t \rightarrow \infty$), and two intermediate times; (c) Sketch heat fluxes as a function of time at the two surfaces; (d) Expression for total energy transferred to wall per unit volume [J/m^3].

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) No internal heat generation.

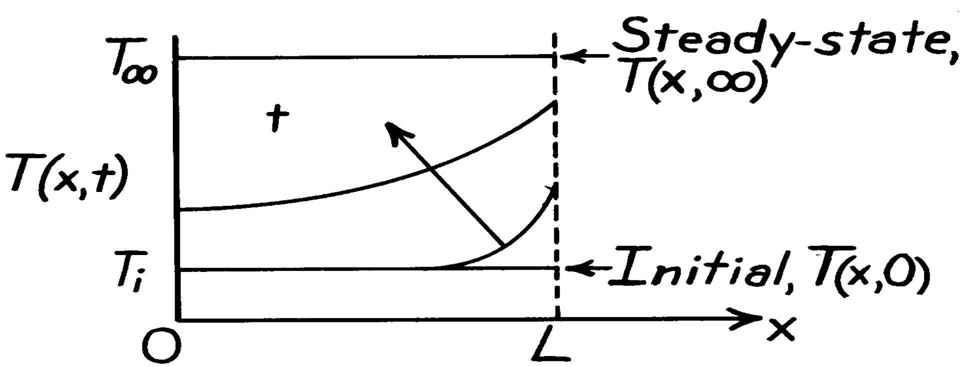
ANALYSIS: (a) For one-dimensional conduction with constant properties, the heat equation has the form,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}$$ <

and the conditions are:

Initial:	$t < 0$	$T(x,0) = T_i$	uniform temperature
Boundaries:	$x = 0$	$\partial T / \partial x\|_0 = 0$	adiabatic surface
	$x = L$	$-k\partial T / \partial x\|_L = h\left[T(L,t) - T_\infty\right]$	surface convection

(b) The temperature distributions are shown on the sketch.



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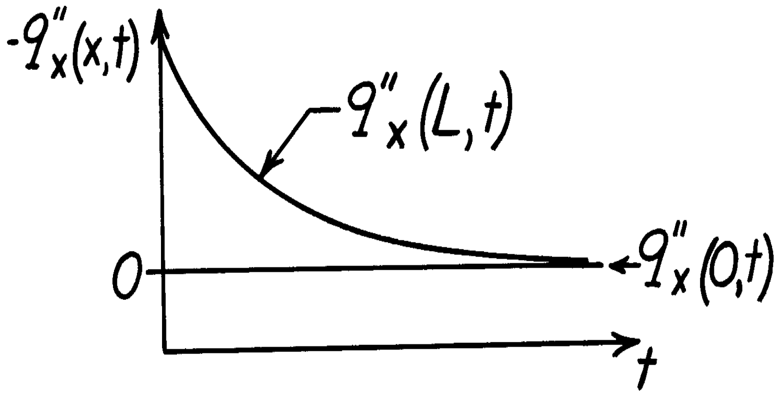
Note that the gradient at $x = 0$ is always zero, since this boundary is adiabatic. Note also that the gradient at $x = L$ decreases with time.

Problem: Thermal Response (cont.)

c) The heat flux, $q''_x(x,t)$, as a function of time, is shown on the sketch for the surfaces $x = 0$ and $x = L$.



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d) The total energy transferred to the wall may be expressed as

$$E_{in} = \int_0^{\infty} q''_{\text{conv}} A_s dt$$

$$E_{in} = hA_s \int_0^{\infty} \left(T_{\infty} - T(L,t)\right) dt$$

Dividing both sides by $A_s L$, the energy transferred per unit volume is

$$\frac{E_{in}}{V} = \frac{h}{L} \int_0^{\infty} \left[T_{\infty} - T(L,t)\right] dt \qquad \left[\text{J/m}^3\right]$$

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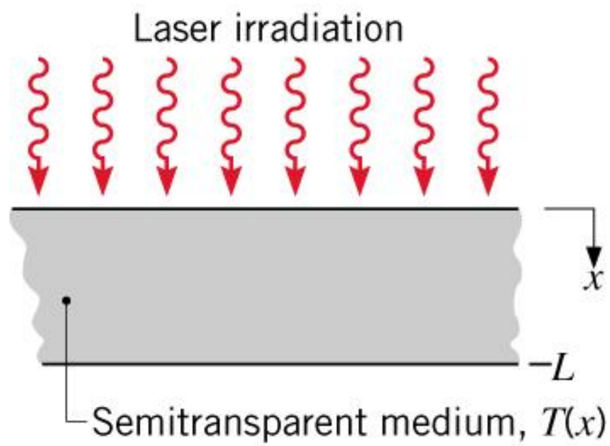
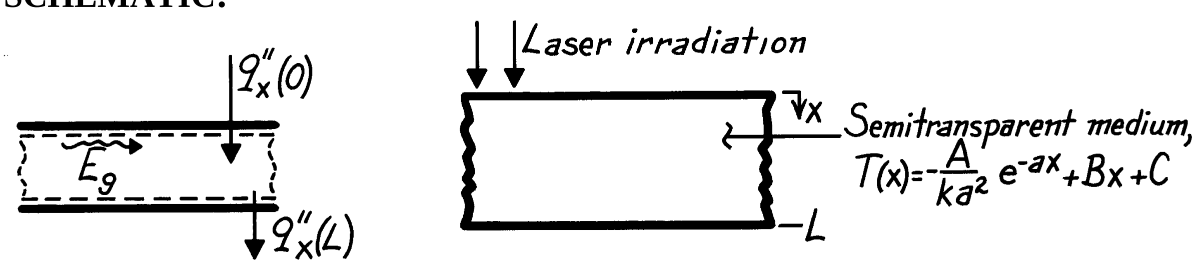
Problem 2.29 Surface heat fluxes, heat generation and total rate of radiation absorption in an irradiated semi-transparent material with a prescribed temperature distribution.



KNOWN: Temperature distribution in a semi-transparent medium subjected to radiative flux.

FIND: (a) Expressions for the heat flux at the front and rear surfaces, (b) The heat generation rate $\dot{q}(x)$, and (c) Expression for absorbed radiation per unit surface area.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in medium, (3) Constant properties, (4) All laser irradiation is absorbed and can be characterized by an internal volumetric heat generation term $\dot{q}(x)$.

ANALYSIS: (a) Knowing the temperature distribution, the surface heat fluxes are found using Fourier's law,

$$q''_x = -k\left[\frac{dT}{dx}\right] = -k\left[\frac{A}{ka}e^{-ax} + B\right]$$

Front Surface, x=0:

$$q''_x(0) = -k\left[\frac{A}{ka} + B\right] = -\left[\frac{A}{a} + kB\right] \qquad <$$

Rear Surface, x=L:

$$q''_x(L) = -k\left[+\frac{A}{ka}e^{-aL} + B\right] = -\left[\frac{A}{a}e^{-aL} + kB\right] \qquad <$$

(b) The heat diffusion equation for the medium is

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) + \frac{\dot{q}}{k} = 0 \quad or \quad \dot{q} = -k\frac{d}{dx}\left(\frac{dT}{dx}\right)$$

$$\dot{q}(x) = -k\frac{d}{dx}\left[+\frac{A}{ka}e^{-ax} + B\right] = Ae^{-ax}. \qquad <$$

(c) Performing an energy balance on the medium,

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_g = 0$$

Problem : Non-uniform Generation (cont.)

On a unit area basis

$$\dot{E}_g'' = -\dot{E}_{\text{in}}'' + \dot{E}_{\text{out}}'' = -q_x''(0) + q_x''(L) = +\frac{A}{a}\left(1 - e^{-aL}\right). \qquad <$$

Alternatively, evaluate $\dot{E}_g''$ by integration over the volume of the medium,

$$\dot{E}_g'' = \int_0^L \dot{q}(x)dx = \int_0^L Ae^{-ax}dx = -\frac{A}{a}\left[e^{-ax}\right]_0^L = \frac{A}{a}\left(1 - e^{-aL}\right).$$